Dense Packings from Algebraic Number Fields and Codes[1]

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Abstract

We introduce a new method from number fields and codes to construct dense packings in the Euclidean spaces. Via the canonical $\mathbb{Q}$-embedding of arbitrary number field K into $\mathbb{R}^{[K:\mathbb{Q}]}$, both the prime ideal $\mathfrak{p}$ and its residue field κ can be embedded as discrete subsets in $\mathbb{R}^{[K:\mathbb{Q}]}$. Thus we can concatenate the embedding image of the Cartesian product of n copies of $\mathfrak{p}$ together with the image of a length n code over κ. This concatenation leads to a packing in the Euclidean space $\mathbb{R}^{n[K:\mathbb{Q}]}$. Moreover, we extend the single concatenation to multiple concatenation to obtain dense packings and asymptotically good packing families. For instance, with the help of Magma, we construct a 256-dimensional packing denser than the Barnes-Wall lattice BW_{256}.

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1 Introduction

The classical problem of packing non-overlapping equal spheres densely in an n-dimensional Euclidean space has attracted the interest of numerous mathematicians for centuries. Many methods and results from different disciplines, such as discrete geometry, combinatorics, number theory and coding theory, etc. have been involved in this problem, while some explicit fascinating dense constructions and asymptotically good packing families have been found. For a detailed survey on the development in this field, the reader may refer to the books of Cassels [6], Conway and Sloane [8], Zong [20]. If the centers of the packed spheres form a discrete additive subgroup of $\mathbb{R}^n$ (lattice), we call it a lattice packing.

Sphere packings are continuous analogues of error-correcting codes in the Hamming space [9]. The basic problem of error-correcting codes is to seek the maximum size of a code given length, alphabet size and minimal Hamming distance, in other words, dense packing of points in Hamming space such that each pair of distinct points is separated at least by the minimum Hamming distance. In the digital world, error-correcting codes are widely employed in information storage and transmission, for example, the blue ray storage format and the communication between space stations and the Earth. Based on the similarities between sphere packings and error-correcting codes, the results in sphere packings can potentially contribute to the development of error-correcting codes.

Meanwhile, some constructions of dense lattice or non-lattice packings are inspired by constructions in coding theory. For example, similar to concatenated codes, Leech and Sloane's "Construction A" method concatenated certain binary codes together with $2 \cdot \mathbb{Z}^n$ to construct new lattice packings (see details in [8, Chapter 5]). Bachoc [2] generalized the method to construct modular lattices using codes over finite involution algebras. In Construction C [20, Chapter 5], the binary expansion of the coordinates in $\mathbb{Z}^n$ is considered. A point is a packing center if and only if its first ℓ coordinate arrays are codewords in certain ℓ binary codes respectively.

Let $\omega = \frac{-1+\sqrt{-3}}{2}$. Xing [19] further investigated the concatenating method. Instead of packings in $\mathbb{Z}^n$, he considered the packings in $\mathcal{O}_K^n$, where $\mathcal{O}_K = \mathbb{Z}[\omega]$ denotes the ring of integers in the number field $\mathbb{Q}(\sqrt{-3})$. That is, for a non-zero prime ideal $\mathfrak{P}$ of $\mathcal{O}_K$, $\mathfrak{P}^n$ can be embedded as a lattice L in $\mathbb{R}^{2n}$ via the canonical

[1]This research began when the author was a PhD candidate at Nanyang Technological University.

$\mathbb{Q}$-embedding of the special number field $\mathbb{Q}(\sqrt{-3})$ into $\mathbb{R}^2$. Hence any subset $\mathcal{P}$ of $\mathfrak{P}^n$ can be regarded as a packing in $\mathbb{R}^{2n}$. Then he replaced the binary expansion in Construction C by the $\mathfrak{P}$-adic expansion, and concatenated the lattice L with some special codes over $\mathcal{O}_K/\mathfrak{P}$. This method produces several dense packings in low dimensions attaining the best-known densities and also obtains an unconditional bound on the asymptotic density exponent $\lambda \geqslant -1.265$ (see [19]). Cheng [7] applied this concatenation to multiplicative lattices and improved the asymptotic density of packing families derived from multiplicative lattices. One natural question is whether we can use arbitrary number field instead of the special quadratic number field $\mathbb{Q}(\sqrt{-3})$ to generalize the constructing method.

In this paper, we extend Xing's method to a general level, i.e. we employ the ideals in $\mathcal{O}_K$ instead of $\mathbb{Z}[\omega]$, where $\mathcal{O}_K$ denotes the ring of integers in an arbitrary number field K. Suppose the extension degree of K over $\mathbb{Q}$ is m. Minkowski interpreted the elements in K as points in the m-dimensional Euclidean space $\mathbb{R}^m$. The interpretation is called "Minkowski Theory" in algebraic number theory (see [14, Section I.5] or [18, Section 5.3]).

By Minkowski's interpretation, we can use the canonical $\mathbb{Q}$-embedding to construct lattices in $\mathbb{R}^m$ from ideals in $\mathcal{O}_K$. In this way, for a non-zero prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, the Cartesian product of n copies of $\mathfrak{p}$ can be embedded as a lattice in $\mathbb{R}^{mn}$. Meanwhile, the codes defined over $\mathcal{O}_K/\mathfrak{p}$ with length n can also be embedded as finite subsets of $\mathbb{R}^{mn}$. Then we can proceed with the concatenating method as Construction A on these two kinds of subsets of $\mathbb{R}^{mn}$ to construct dense packings. For instance, we construct a 256-dimensional packing with center density δ satisfying $\log_2 \delta \geqslant 208.09$, which is larger than 192 of the Barnes-Wall lattice BW_{256} (see the table of dense packings in [8, Table 1.3]). Furthermore, for different choices of number fields and prime ideals, we also provide several asymptotically good packing families.

In section 2, we recall the formal definitions and necessary properties of sphere packing densities, algebraic number fields and codes. They play important roles in the main results. In section 3, we describe our generalized concatenating method in detail. Several examples of dense packings and asymptotically good packing families for different choices of number fields and prime ideals are provided. The detailed numerical results are provided in Tables 1 - 4. In Section 4, we conclude our main contribution.

2 Preliminaries

2.1 Sphere Packing

Let $\mathcal{P}$ be the set of centers of packed spheres and $\mathcal{B}_N(R)$ be the set

$$\left\{\mathbf{v} = (a_1, \cdots, a_N) \in \mathbb{R}^N : \|\mathbf{v}\| = \sqrt{a_1^2 + \cdots + a_N^2} \leqslant R\right\},$$

where $\|\mathbf{v}\|$ denotes the Euclidean length of vector $\mathbf{v}$. As a sphere packing construction is determined by the arrangement of the sphere centers, we just use $\mathcal{P}$ to denote the corresponding packing.

For a packing $\mathcal{P}$, the radius of the equal packed spheres is $r = d_E(\mathcal{P})/2$, where $d_E(\mathcal{P})$ is the minimum Euclidean distance between two distinct points in $\mathcal{P}$. Then the density $\Delta(\mathcal{P})$ of packing $\mathcal{P}$ is defined as

$$\Delta(\mathcal{P}) = \limsup_{R\to\infty} \frac{|\mathcal{P} \cap \mathcal{B}_N(R)| \cdot r^N \cdot V_N}{\mathrm{vol}(\mathcal{B}_N(R+r))},$$

where V_N is the volume of the unit sphere in $\mathbb{R}^N$, that is

$$V_N = \begin{cases} \dfrac{\pi^{N/2}}{(N/2)!}, & \text{if } N \text{ is even;} \\ \dfrac{2^N \pi^{(N-1)/2}\left((N-1)/2\right)!}{N!}, & \text{if } N \text{ is odd.} \end{cases}$$

The sphere packing problem is to construct packings obtaining large density $\Delta(\mathcal{P})$. Moreover, the center density $\delta(\mathcal{P})$ and density exponent $\lambda(\mathcal{P})$ are defined respectively as

$$\delta(\mathcal{P}) = \frac{\Delta(\mathcal{P})}{V_N}, \qquad \lambda(\mathcal{P}) = \frac{1}{N}\log_2 \Delta(\mathcal{P}).$$

If $\mathcal{P} = L$ forms a lattice, the density of lattice packing L can be simplified as

$$\Delta(L) = \frac{(d_E(L)/2)^N V_N}{\det(L)},$$

where $\det(L)$ is the determinant of L. Note that from Minkowski's Convex Body Theorem (see [13, Theorem 1.4] or [15, Theorem 4]), for any rank N lattice L, the minimum Euclidean distance of L satisfies

$$d_E(L) \leqslant \sqrt{N}\det(L)^{1/N}. \tag{1}$$

When we explore the asymptotic behavior of a packing family $\mathcal{F} = \left\{\mathcal{P}^{(N)}\right\}$ as the dimension N of packing $\mathcal{P}^{(N)}$ tends to ∞, we consider the asymptotic density exponent of the family

$$\lambda(\mathcal{F}) = \limsup_{N\to\infty} \frac{1}{N}\log_2 \Delta\left(\mathcal{P}^{(N)}\right). \tag{2}$$

Note that by Stirling formula, as $N \to \infty$, we have

$$\log_2 V_N = -\frac{N}{2}\log_2\frac{N}{2\pi e} - \frac{1}{2}\log_2(N\pi) - \epsilon, \tag{3}$$

where $0 < \epsilon < (\log_2 e)/(6N)$.

For the asymptotic aspect, Minkowski gave a nonconstructive bound that asserts that there exists some packing family $\mathcal{F}$ such that the asymptotic density exponent $\lambda(\mathcal{F})$ satisfies $\lambda(\mathcal{F}) \geqslant -1$ (see [6, p.184]). It is of interest to construct families with $\lambda(\mathcal{F}) > -\infty$ explicitly (such families are called *asymptotically good*). Known constructive bounds for families with polynomial or exponential construction complexity in terms of dimension N are listed in the book of Litsyn and Tsfasman [18, p.628]. Rush [17] proved that Minkowski's bound on asymptotic density $\lambda \geqslant -1$ can be attained by lattice packings constructed from codes. More recently, Gaborit and Zémor [10] improved the density by a linear factor to the quantity of the form $cn2^{-n}$ for constant c.

2.2 Algebraic Number Fields

Let K be an algebraic number field of degree m over $\mathbb{Q}$, and let $\mathcal{O}_K$ be its ring of integers. Suppose K has s real embeddings

$$\rho_1, \cdots, \rho_s : K \hookrightarrow \mathbb{R},$$

and t pairs of complex conjugate embeddings

$$\sigma_1, \overline{\sigma}_1, \cdots, \sigma_t, \overline{\sigma}_t : K \hookrightarrow \mathbb{C}.$$

Thus $m = s + 2t$. We consider the canonical embedding $\tau : K \hookrightarrow \mathbb{R}^{s+2t}$, that is, for any $\alpha \in K$, $\tau(\alpha)$ is the following vector

$$\left(\rho_1(\alpha), \cdots, \rho_s(\alpha), \sqrt{2}\Re\sigma_1(\alpha), \sqrt{2}\Im\sigma_1(\alpha), \cdots, \sqrt{2}\Re\sigma_t(\alpha), \sqrt{2}\Im\sigma_t(\alpha)\right), \tag{4}$$

where $\Re$ denotes the real part and $\Im$ denotes the imaginary part of a complex number. τ can be directly extended to $K^n \hookrightarrow \mathbb{R}^{mn}$, which is also denoted by τ in this paper without causing any confusion.

Note that in the software Magma V2.21-4 [4, 5], the embedding τ defined above is called the Minkowski map and it is employed to compute the "minimum norm" (square of minimum Euclidean distance) of a

Minkowski lattice. In Subsection 3.4, we will use the software to compute the density of some explicit examples from our construction.

Set $\mathrm{Tr} = \mathrm{Trace}_{K/\mathbb{Q}}$ and $\mathrm{N} = \mathrm{Norm}_{K/\mathbb{Q}}$. By the Minkowski Theory [14, Section I.5], the embedding τ maps the non-zero ideals of $\mathcal{O}_K$ to some special lattices in $\mathbb{R}^m$. The following lemma characterizes the determinant of such lattices.

Lemma 2.1 ([14, Section I.5])**.** *For any non-zero ideal $I \subseteq \mathcal{O}_K$, $L_I := \tau(I)$ is a lattice of rank m. The determinant of L_I is*

$$\det L_I = \mathcal{N}(I)\sqrt{|D_K|},$$

where D_K is the discriminant of K and $\mathcal{N}(I) = [\mathcal{O}_K : I]$ is the absolute norm of I.

Generally it remains hard to compute the minimum Euclidean distance of lattices. However, we can estimate a lower bound on the Euclidean length of non-zero points in the lattice L_I. The proof is sketched here and a similar discussion on general ideal lattices can be found in [3]. Note that in [3], the distance between two points is defined using some special quadratic forms, while in this paper, we only focus on the standard Euclidean distance, which refers to the original meaning of packing in the Euclidean space and brings convenience in computation (Magma uses standard Euclidean distance as built-in measure).

Lemma 2.2. *Let $I \subseteq \mathcal{O}_K$ be a non-zero ideal. For any $0 \neq \alpha \in I$, the Euclidean length of the vector $\tau(\alpha) \in L_I$ satisfies*

$$\|\tau(\alpha)\| \geqslant \sqrt{m} \cdot |\mathrm{N}(\alpha)|^{1/m} \geqslant \sqrt{m} \cdot \mathcal{N}(I)^{1/m}.$$

In other words, $d_E(L_I) \geqslant \sqrt{m} \cdot \mathcal{N}(I)^{1/m}$.

Proof. The Euclidean length of the vector $\tau(\alpha)$ satisfies

$$\begin{aligned}
\|\tau(\alpha)\|^2 &= \sum_{i=1}^{s} \rho_i(\alpha)^2 + 2\sum_{j=1}^{t} \sigma_j(\alpha)\bar{\sigma}_j(\alpha) \\
&\geqslant (s+2t)\left[\prod_{i=1}^{s} \rho_i(\alpha)^2 \cdot \prod_{j=1}^{t} \sigma_j(\alpha)\overline{\sigma}_j(\alpha) \cdot \prod_{j=1}^{t} \sigma_j(\alpha)\overline{\sigma}_j(\alpha)\right]^{\frac{1}{s+2t}} \\
&= m\left[(\mathrm{N}(\alpha))^2\right]^{1/m} \geqslant m \cdot \mathcal{N}(I)^{2/m}.
\end{aligned}$$

The last $\geqslant$ becomes an equality if and only if the ideal I is a principal ideal generated by α.

□

The following lemma is a basic fact in algebraic number theory (see [14, Section I.2]).

Lemma 2.3. *For any non-zero element $\beta \in \mathcal{O}_K$, we have $|\mathrm{N}(\beta)| \geqslant 1$.*

From the above lemmas, the minimum Euclidean length of non-zero elements in $L_{\mathcal{O}_K}$ satisfies $d_E(L_{\mathcal{O}_K}) \geqslant \sqrt{m}$. Indeed, as the vector $\tau(1)$ has Euclidean length $\sqrt{m}$, we have

$$d_E(L_{\mathcal{O}_K}) = \sqrt{m}. \tag{5}$$

2.3 Coding Theory

We recall some notations and results in coding theory.

For a q-ary code C, let $n(C), M(C)$ and $d_H(C)$ denote the length, the size, and the minimum Hamming distance of C, respectively. Such a code is usually referred to as an $(n(C), M(C), d_H(C))$-code. Moreover, the relative minimum distance $\varrho(C)$ and the rate $R(C)$ are defined respectively as

$$\varrho(C) = \frac{d_H(C)}{n(C)}, \quad R(C) = \frac{\log_q M(C)}{n(C)}.$$

In particular, if a code C forms a linear space over $\mathbb{F}_q$, then the code C is called an $[n(C), k(C), d_H(C)]$-linear code over $\mathbb{F}_q$, where $k(C) := \log_q M(C)$ is called the dimension of C. In this case, the rate $R(C) = \frac{k(C)}{n(C)}$.

Let U_q be the set of the ordered pair $(\varrho, R) \in \mathbb{R}^2$, for which there exists a family $\{C_i\}_{i=0}^{\infty}$ of q-ary codes such that $n(C_i)$ increasingly goes to ∞ as i tends to ∞ and

$$\varrho = \lim_{i \to \infty} \varrho(C_i), \quad \text{and} \quad R = \lim_{i \to \infty} R(C_i).$$

Here is a result on U_q:

Proposition 2.4 ([18, Section 1.3.1] or [19, Proposition 3.1])**.** *There exists a continuous function $R_q(\varrho)$, $\varrho \in [0,1]$, such that*

$$U_q = \left\{(\varrho, R) \in \mathbb{R}^2 : 0 \leqslant R \leqslant R_q(\varrho),\ 0 \leqslant \varrho \leqslant 1\right\}.$$

Moreover, $R_q(0) = 1$, $R_q(\varrho) = 0$ for $\varrho \in [(q-1)/q, 1]$, and $R_q(\varrho)$ decreases on the interval $[0, (q-1)/q]$.

For $0 < \varrho < 1$, the q-ary entropy function is given as

$$H_q(\varrho) = \varrho \log_q(q-1) - \varrho \log_q \varrho - (1-\varrho)\log_q(1-\varrho).$$

The asymptotic Gilbert-Varshamov (GV) bound indicates that

$$R_q(\varrho) \geqslant R_{GV}(q, \varrho) := 1 - H_q(\varrho), \quad \text{for all } \varrho \in \left(0, \frac{q-1}{q}\right). \tag{6}$$

Moreover, for any given rate R, there exists a family of linear codes which meets the GV bound (see [12, Section 17.7]).

3 Main Results

In this section, we introduce our new construction of dense sphere packings. In particular, several explicit constructions and asymptotically good packing families are provided at the end of this section. Our idea is to apply special concatenating methods on L_I defined in Lemma 2.1.

Let $[K : \mathbb{Q}] = m$ and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$. Assume that the residue field $F_{\mathfrak{p}} = \mathcal{O}_K/\mathfrak{p}$ of $\mathfrak{p}$ is isomorphic to the finite field $\mathbb{F}_q$. Let τ be the canonical embedding defined in Eq. (4).

For $i \in \mathbb{N}$, let $L_{\mathfrak{p}^i}^n$ denote the Cartesian product of n identical copies of lattice $L_{\mathfrak{p}^i} = \tau(\mathfrak{p}^i)$. Then $L_{\mathfrak{p}^i}^n$ is a lattice of rank mn by Lemma 2.1. Moreover, the determinant satisfies

$$\det L_{\mathfrak{p}^i}^n = \left(\det L_{\mathfrak{p}^i}\right)^n,$$

which is straightforward from the definition of lattice determinant. In addition, by the definition of $L_{\mathfrak{p}^i}^n$, the minimum Euclidean distance satisfies

$$d_E(L_{\mathfrak{p}^i}^n) = d_E(L_{\mathfrak{p}^i}).$$

For simplicity, in this section we write $L_i = L_{\mathfrak{p}^i}$ for short.

As $\mathcal{O}_K$ is a Dedekind domain (see [14, Section I.3] or [1, Chapter 9]), let $\kappa = \mathcal{O}_K/\mathfrak{p}$, then we have $\dim_\kappa \mathfrak{p}^i/\mathfrak{p}^{i+1} = 1$ for all $i \in \mathbb{N}$. We know $\kappa \cong \mathbb{F}_q$. Thus for $i \in \mathbb{N}$, we can choose the sets $S_i := \left\{\alpha_1^{(i)} = 0, \alpha_2^{(i)}, \cdots, \alpha_q^{(i)}\right\} \subseteq \mathfrak{p}^i$ such that

$$\alpha_1^{(i)} \bmod \mathfrak{p}^{i+1}, \cdots, \alpha_q^{(i)} \bmod \mathfrak{p}^{i+1} \tag{7}$$

represent the q distinct elements in $\mathfrak{p}^i/\mathfrak{p}^{i+1}$.

3.1 Concatenation with One Code

We fix one index $i_0 \in \mathbb{N}_{\geqslant 1}$ in the discussion here and generalize the result to a family of indices in the next subsection. Let $\mathcal{P}$ be a subset of $L_{i_0}^n$, which can be regarded as a packing in $\mathbb{R}^{mn}$.

Given a q-ary code C with length n, in order to perform our concatenation, we take the code alphabet set of C to be $S_{i_0-1} \subseteq \mathfrak{p}^{i_0-1}$ (see Eq. (7)). In this way, C can be regarded as a finite subset of $\mathcal{O}_K^n$. The advantage of using S_{i_0-1} as the alphabet set is that it can help us bound the packing radius of our construction. The details are included in the proof of Proposition 3.2.

We define $\tau(\mathfrak{c})$ for each codeword $\mathfrak{c} = (c_1, \cdots, c_n) \in C \subseteq \mathcal{O}_K^n$ as

$$\tau(\mathfrak{c}) := (\tau(c_1), \cdots, \tau(c_n)) \in \mathbb{R}^{mn},$$

and take $\tau(C) := \{\tau(\mathfrak{c}) : \mathfrak{c} \in C\}$. We consider the concatenation

$$\tau(C) + \mathcal{P} := \{\mathbf{a} + \mathbf{p} \in \mathbb{R}^{mn} : \mathbf{a} \in \tau(C), \mathbf{p} \in \mathcal{P}\},$$

which corresponds to a packing in $\mathbb{R}^{mn}$. We will analyze the density of such a packing in this subsection.

The following lemma characterizes the minimum Euclidean distance of the concatenation $\tau(C)+\mathcal{P}$, which will play an important role in later discussions.

Lemma 3.1. *Let $\mathfrak{p}$ be a non-zero prime ideal in $\mathcal{O}_K$ with absolute norm $\mathcal{N}(\mathfrak{p}) = q$ (i.e., the residue field $F_\mathfrak{p} \cong \mathbb{F}_q$). For $i_0 \in \mathbb{N}_{\geqslant 1}$, let*

(i) $\mathcal{P}$ be a subset of $L_{i_0}^n$,

(ii) C be a q-ary (n, M, d_C)-code over the alphabet set S_{i_0-1}. In addition, C contains the zero codeword.

Then the minimum Euclidean distance of the packing $\tau(C) + \mathcal{P}$ satisfies

$$d_E(\tau(C) + \mathcal{P}) \geqslant \min\left\{d_E(L_{i_0-1})\sqrt{d_C},\ d_E(\mathcal{P})\right\}.$$

In particular, if $d_E(L_{i_0-1})\sqrt{d_C} \geqslant d_E(\mathcal{P})$, we have exactly

$$d_E(\tau(C) + \mathcal{P}) = d_E(\mathcal{P}).$$

Proof. Let $\tau(\mathfrak{c}) + \mathbf{p}$ be a non-zero element in $\tau(C) + \mathcal{P}$, where $\mathfrak{c} \in C$ and $\mathbf{p} \in \mathcal{P}$. If $\mathfrak{c} = \mathbf{0}$, then

$$\|\tau(\mathfrak{c}) + \mathbf{p}\| = \|\mathbf{p}\| \geqslant d_E(\mathcal{P}).$$

If $\mathfrak{c} \neq \mathbf{0}$, without loss of generality, we assume $\mathfrak{c} = (c_1, \cdots, c_e, \mathbf{0})$, where e is the Hamming weight of $\mathfrak{c}$ and $c_i \neq 0$ for $1 \leqslant i \leqslant e$. As $\mathcal{P} \subseteq L_{i_0}^n$, we can further assume $\mathbf{p} = (\tau(\rho_1), \cdots, \tau(\rho_n))$, where $\rho_j \in \mathfrak{p}^{i_0}$ for $1 \leqslant j \leqslant n$. Thus

$$\tau(\mathfrak{c}) + \mathbf{p} = (\tau(c_1 + \rho_1), \cdots, \tau(c_e + \rho_e), \tau(\rho_{e+1}), \cdots, \tau(\rho_n)).$$

As $c_1, \cdots, c_e$ are non-zero elements in S_{i_0-1}, i.e., for $1 \leqslant i \leqslant e$, $c_i \in \mathfrak{p}^{i_0-1} \setminus \mathfrak{p}^{i_0}$, we have that $c_i + \rho_i$ are non-zero elements in $\mathfrak{p}^{i_0-1}$ for $1 \leqslant i \leqslant e$. Thus

$$\|\tau(\mathfrak{c}) + \mathbf{p}\|^2 \geqslant \sum_{i=1}^{e} \|\tau(c_i + \rho_i)\|^2 \geqslant e \cdot d_E^2(L_{i_0-1}) \geqslant d_C \cdot d_E^2(L_{i_0-1}).$$

In summary, the minimum Euclidean distance of the packing $\tau(C) + \mathcal{P}$ satisfies

$$d_E\left(\tau(C) + \mathcal{P}\right) \geqslant \min\left\{ d_E(L_{i_0-1})\sqrt{d_C},\ d_E(\mathcal{P}) \right\}. \tag{8}$$

In particular, if $d_E(L_{i_0-1})\sqrt{d_C} \geqslant d_E\left(\mathcal{P}\right)$, then from Eq. (8) above, we have the minimum distance $d_E\left(\tau(C) + \mathcal{P}\right) \geqslant d_E(\mathcal{P})$. Meanwhile, as C contains the zero codeword, $\mathcal{P}$ is a subset of $\tau(C) + \mathcal{P}$, thus $d_E(\mathcal{P}) \geqslant d_E\left(\tau(C) + \mathcal{P}\right)$. Finally we have exactly

$$d_E\left(\tau(C) + \mathcal{P}\right) = d_E(\mathcal{P}).$$

□

The following proposition generalizes Proposition 2.3 of [19], where only the special case $K = \mathbb{Q}(\sqrt{-3})$ is discussed.

Proposition 3.2. *Under the same assumption on $\mathcal{P}$ and C as in Lemma 3.1,*

(i) if $d_E(L_{i_0-1})\sqrt{d_C} \geqslant d_E\left(\mathcal{P}\right)$, then the density of $\tau(C) + \mathcal{P}$ as a packing in $\mathbb{R}^{mn}$ is at least $M \cdot \Delta\left(\mathcal{P}\right)$, where $\Delta\left(\mathcal{P}\right)$ is the density of the packing $\mathcal{P}$ as a packing in $\mathbb{R}^{mn}$, and M is the size of the code C;

(ii) if $\mathcal{P}$ is a lattice and C satisfies that for any $\mathfrak{u}, \mathfrak{v} \in C$, the sum $\tau(\mathfrak{u}) + \tau(\mathfrak{v})$ is equal to $\tau(\mathfrak{w}) + \mathbf{p}$ for some $\mathfrak{w} \in C$ and $\mathbf{p} \in \mathcal{P}$, then $\tau(C) + \mathcal{P}$ is also a lattice.

Proof. (i) We denote the volume of the unit sphere in $\mathbb{R}^N$ by V_N and the sphere of radius b by $\mathcal{B}_N(b)$. Let $s = \max\{\|\tau(\mathfrak{c})\| : \mathfrak{c} \in C\}$. For any $\mathfrak{c} \in C$ and $\mathbf{p} \in \mathcal{P} \cap \mathcal{B}_{mn}(b)$, we have $\tau(\mathfrak{c}) + \mathbf{p} \in (\tau(\mathfrak{c}) + \mathcal{P}) \cap \mathcal{B}_{mn}(b+s)$. This implies that

$$\left|(\tau(\mathfrak{c}) + \mathcal{P}) \cap \mathcal{B}_{mn}(b+s)\right| \geqslant \left|\mathcal{P} \cap \mathcal{B}_{mn}(b)\right|.$$

Moreover, as the elements in the alphabet set S_{i_0-1} of C represent the q distinct elements in $\mathfrak{p}^{i_0-1}/\mathfrak{p}^{i_0}$ and $\mathcal{P} \subseteq L_{i_0}^n$, we have if $\mathfrak{c}_i \neq \mathfrak{c}_j$ then

$$(\tau(\mathfrak{c}_i) + \mathcal{P}) \cap (\tau(\mathfrak{c}_j) + \mathcal{P}) = \emptyset.$$

We write $d = d_E\left(\tau(C) + \mathcal{P}\right)$ for short and from Lemma 3.1 we have $d = d_E(\mathcal{P})$. Finally we obtain

$$\begin{aligned}
& \Delta\left(\tau(C) + \mathcal{P}\right) \\
= \ & \limsup_{b \to \infty} \frac{\left|(\tau(C) + \mathcal{P}) \cap \mathcal{B}_{mn}\left(b+s\right)\right| (d/2)^{mn} V_{mn}}{\mathrm{vol}(\mathcal{B}_{mn}(b+s+d/2)} \\
= \ & \limsup_{b \to \infty} \frac{\left(\sum_{\mathfrak{c} \in C} \left|(\tau(\mathfrak{c}) + \mathcal{P}) \cap \mathcal{B}_{mn}\left(b+s\right)\right|\right) (d/2)^{mn} V_{mn}}{\mathrm{vol}(\mathcal{B}_{mn}(b+s+d/2)} \\
\geqslant \ & \limsup_{b \to \infty} \frac{|C| \cdot \left|\mathcal{P} \cap \mathcal{B}_{mn}(b)\right| \left(d_E(\mathcal{P})/2\right)^{mn} V_{mn}}{\mathrm{vol}\left(\mathcal{B}_{mn}\left(b+s+d/2\right)\right)} \\
= \ & |C| \cdot \Delta\left(\mathcal{P}\right) = M \cdot \Delta(\mathcal{P}).
\end{aligned}$$

(ii) By the definition of a lattice, this part is true. □

3.2 Concatenation with a Family of Codes

For a family of q-ary codes $\{C_i\}_{i=0}^{\ell-1}$, where $\ell \in \mathbb{N}_{\geqslant 1}$, we can take the code alphabet set of C_i to be S_i (see Eq. (7)). Note that the choice of the alphabet set of the codes is only used in the proof. For the computation, we only care about the length, size and minimum Hamming distance of the codes.

Similar to Subsection 3.1, for any subset $\mathcal{P}$ of L_ℓ^n, and a family of q-ary codes $\mathcal{C} = \{C_i\}_{i=0}^{\ell-1}$ with length n, we consider the concatenation

$$\tau(\mathcal{C}) + \mathcal{P} := \sum_{i=0}^{\ell-1} \tau(C_i) + \mathcal{P} = \left\{ \sum_{i=0}^{\ell-1} \mathbf{a}_i + \mathbf{p} \in \mathbb{R}^{mn} : \mathbf{a}_i \in \tau(C_i), \mathbf{p} \in \mathcal{P} \right\}. \tag{9}$$

Note that for each $0 \leqslant i \leqslant \ell - 1$, as the code alphabet set of C_i is S_i, the set $\tau(C_i)$ is a finite subset of L_i^n. Meanwhile $\mathcal{P}$ is a subset of L_ℓ^n. As for any $0 \leqslant i \leqslant \ell$, the lattice L_i^n is a sublattice of $L_{\mathcal{O}_K}^n$, the concatenation Eq. (9) makes sense within $L_{\mathcal{O}_K}^n$.

Proposition 3.3. *Let $\mathfrak{p}$ be a non-zero prime ideal in $\mathcal{O}_K$ with absolute norm $\mathcal{N}(\mathfrak{p}) = q$. For $\ell \in \mathbb{N}_{\geqslant 1}$, let*

(i) $\mathcal{P}$ be a subset of L_ℓ^n,

(ii) $\mathcal{C} = \{C_i\}_{i=0}^{\ell-1}$ be a family of q-ary codes, where C_i is an (n, M_i, d_{C_i})-code, the alphabet set of C_i is S_i, and $d_{C_i} \geqslant \left\lceil \frac{d_E^2(\mathcal{P})}{d_E^2(L_i)} \right\rceil$. In addition, for each $0 \leqslant i \leqslant \ell - 1$, C_i contains the zero codeword.

Then the density of $\tau(\mathcal{C}) + \mathcal{P}$ as a packing in $\mathbb{R}^{mn}$ is at least $\Delta(\mathcal{P}) \cdot \prod_{i=0}^{\ell-1} M_i$.

Proof. Let $\mathcal{P}_\ell := \mathcal{P} \subseteq L_\ell^n$. For i from $\ell - 1$ to 0, we can recursively define

$$\mathcal{P}_i := \tau(C_i) + \mathcal{P}_{i+1} \subseteq L_i^n.$$

Note that $\mathcal{P}_0 = \tau(\mathcal{C}) + \mathcal{P}$.

We claim that for $0 \leqslant i \leqslant \ell - 1$,

$$d_E(\mathcal{P}_i) = d_E(\mathcal{P}_{i+1}) \quad \text{and} \quad \Delta(\mathcal{P}_i) \geqslant M_i \cdot \Delta(\mathcal{P}_{i+1}). \tag{10}$$

We use induction on $k = \ell - i$ to prove the claim. When $k = 1$, $i = \ell - 1$, as the minimum Hamming distance of $C_{\ell-1}$ satisfies $d_{C_{\ell-1}} d_E^2(L_{\ell-1}) \geqslant d_E^2(\mathcal{P}) = d_E^2(\mathcal{P}_\ell)$, by Lemma 3.1 and Proposition 3.2 (i), we have

$$d_E(\mathcal{P}_{\ell-1}) = d_E(\tau(C_{\ell-1}) + \mathcal{P}_\ell) = d_E(\mathcal{P}_\ell) \quad \text{and} \quad \Delta(\mathcal{P}_{\ell-1}) \geqslant M_{\ell-1} \cdot \Delta(\mathcal{P}_\ell).$$

Suppose for all k in the range $1 \leqslant k < k_0 + 1 \leqslant \ell$, the statement Eq. (10) is true, i.e., $i = \ell - k$,

$$d_E(\mathcal{P}_{\ell-k}) = d_E(\mathcal{P}_{\ell-k+1}) \quad \text{and} \quad \Delta(\mathcal{P}_{\ell-k}) \geqslant M_{\ell-k} \cdot \Delta(\mathcal{P}_{\ell-k+1}) \tag{11}$$

is true. By induction, we need to prove

$$d_E(\mathcal{P}_{\ell-k_0-1}) = d_E(\mathcal{P}_{\ell-k_0}) \quad \text{and} \quad \Delta(\mathcal{P}_{\ell-k_0-1}) \geqslant M_{\ell-k_0-1} \cdot \Delta(\mathcal{P}_{\ell-k_0}).$$

From Eq. (11), we have

$$d_E(\mathcal{P}_{\ell-k_0}) = d_E(\mathcal{P}_{\ell-k_0+1}) = \cdots = d_E(\mathcal{P}_{\ell-1}) = d_E(\mathcal{P}_\ell) = d_E(\mathcal{P}).$$

Besides, the minimum Hamming distance of $C_{\ell-k_0-1}$ satisfies

$$d_{C_{\ell-k_0-1}} d_E^2(L_{\ell-k_0-1}) \geqslant d_E^2(\mathcal{P}) = d_E^2(\mathcal{P}_{\ell-k_0}).$$

Then by Lemma 3.1 and Proposition 3.2 (i), as $\mathcal{P}_{\ell-k_0-1} = \tau(C_{\ell-k_0-1}) + \mathcal{P}_{\ell-k_0}$, we get

$$d_E(\mathcal{P}_{\ell-k_0-1}) = d_E(\mathcal{P}_{\ell-k_0}) \quad \text{and} \quad \Delta(\mathcal{P}_{\ell-k_0-1}) \geqslant M_{\ell-k_0-1} \cdot \Delta(\mathcal{P}_{\ell-k_0}).$$

Thus we have proved the claim Eq. (10). From the claim, we easily obtain $\Delta(\mathcal{P}_0) \geqslant \Delta(\mathcal{P}) \cdot \prod_{i=0}^{\ell-1} M_i$. $\square$

Remark 3.4. *From the above proposition, given a dense packing $\mathcal{P}$ from the canonical embedding Eq. (4) on some prime ideal in a certain algebraic number field, we can concatenate several codes satisfying the requirements in Proposition 3.3 to $\mathcal{P}$ to obtain some new denser packings. The density increases by a ratio larger than the product of the sizes of these codes.*

Note that generally it is hard to determine the minimum Euclidean distance of $\mathcal{P} \subseteq L_\ell^n \subseteq \mathbb{R}^{mn}$. However, we can consider special choices of $\mathcal{P}$ with some algebraic structure. The following corollary, which plays a crucial role in Subsection 3.4 for Examples 3.8 - 3.11, considers the case $\mathcal{P} = L_\ell^n$. The advantage is that for all $i \in \mathbb{N}$, we have $d_E(L_i^n) = d_E(L_i)$. Instead of searching for the minimum Euclidean distance of L_i^n in $\mathbb{R}^{mn}$, we can search for the minimum Euclidean distance of L_i in $\mathbb{R}^m$. In our examples, m is small such that Magma can be used to compute the minimum Euclidean distance.

Corollary 3.5. *Let $\mathfrak{p}$ be a non-zero prime ideal in $\mathcal{O}_K$ with absolute norm $\mathcal{N}(\mathfrak{p}) = q$. For $\ell \in \mathbb{N}_{\geqslant 1}$, let $\mathcal{C} = \{C_i\}_{i=0}^{\ell-1}$ be a family of q-ary codes, where C_i is an (n, M_i, d_{C_i})-code, the alphabet set of C_i is S_i, and $d_{C_i} \geqslant \left\lceil \frac{d_E^2(L_\ell)}{d_E^2(L_i)} \right\rceil$. In addition, for each $0 \leqslant i \leqslant \ell - 1$, the code C_i contains the zero codeword.*

Then the density of $\tau(\mathcal{C}) + L_\ell^n$ as a packing in $\mathbb{R}^{mn}$ satisfies

$$\Delta(\tau(\mathcal{C}) + L_\ell^n) \geqslant \frac{(d_E(L_\ell)/2)^{mn}\, V_{mn}}{\left(q^\ell \sqrt{|D_K|}\right)^n} \cdot \prod_{i=0}^{\ell-1} M_i,$$

where D_K is the discriminant of K and V_{mn} is the volume of the unit sphere in $\mathbb{R}^{mn}$. Moreover, the center density $\delta = \delta(\tau(\mathcal{C}) + L_\ell^n)$ satisfies

$$\log_2 \delta \geqslant mn \log_2 d_E(L_\ell) - mn - n\ell \log_2 q - \frac{n}{2} \log_2 |D_K| + \sum_{i=0}^{\ell-1} \log_2 M_i.$$

In particular, if for all $0 \leqslant i \leqslant \ell - 1$, the code C_i is a linear code of dimension $k_i = \log_q M_i$, then

$$\log_2 \delta \geqslant mn \log_2 d_E(L_\ell) - mn - n\ell \log_2 q - \frac{n}{2} \log_2 |D_K| + \log_2 q \cdot \sum_{i=0}^{\ell-1} k_i.$$

Proof. Note that $\mathcal{N}(\mathfrak{p}^\ell) = q^\ell$ and then by Lemma 2.1

$$\det L_\ell^n = (\det L_\ell)^n = \left(q^\ell \sqrt{|D_K|}\right)^n.$$

$\square$

Remark 3.6. *For concatenation with finite codes, we care more about the dimension of the packing as we need to compare the new construction with the known good packings in the Euclidean space with the same dimension. Hence we fix the length n of the codes first. As for $0 \leqslant i \leqslant \ell - 1$, the lattice L_i is a sublattice of L_0, we have $d_E(L_i) \geqslant d_E(L_0)$. Thus the length n satisfies $n \geqslant \left\lceil \frac{d_E^2(L_\ell)}{d_E^2(L_0)} \right\rceil \geqslant \lceil q^{2\ell/m} \rceil$ by Eq. (5) and Lemma 2.2. Therefore, we can concatenate at most $\ell = \left\lfloor \frac{m}{2} \log_q n \right\rfloor$ codes to L_ℓ^n. This bound will be used in the computation of Subsection 3.4.*

3.3 Asymptotic Properties

In this subsection, we show that our construction will lead to asymptotically good packing families, which means we can obtain several constructive bounds on the asymptotic density exponent Eq. (2).

We will employ families of linear codes that meet the GV bound. As there exist only exponential time algorithms or randomized polynomial algorithms to construct such families (see [12, Section 17.7]), our corresponding asymptotic density exponent bounds are exponential constructive bounds.

From Eq. (1) and Lemma 2.2, for $0 \leqslant i \leqslant \ell - 1$, we know

$$\frac{d_E^2(L_\ell)}{d_E^2(L_i)} \leqslant q^{2(\ell-i)/m} |D_K|^{1/m}. \tag{12}$$

Different from the finite concatenation in Subsection 3.2, for asymptotic results, we first fix ℓ and construct the packing for each $\ell \in \mathbb{N}_{\geqslant 1}$. The length n_ℓ of the codes is set depending on ℓ.

We uniformly set $n_\ell = \lceil q^{2\ell/m} |D_K|^{1/m} \rceil$. Then by Eq. (12), $n_\ell \geqslant \left\lceil \frac{d_E^2(L_\ell)}{d_E^2(L_i)} \right\rceil$ for all $0 \leqslant i \leqslant \ell - 1$. Based on the GV bound Eq. (6), for $0 \leqslant i \leqslant \ell - 1$, we can choose q-ary

$$\left[n_\ell, k_i^{(n_\ell)}, \left\lceil \frac{d_E^2(L_\ell)}{d_E^2(L_i)} \right\rceil \right]$$

linear code $C_i^{(n_\ell)}$ such that the rate is

$$\frac{k_i^{(n_\ell)}}{n_\ell} \geqslant R_{GV}(q, \varrho_i^{(n_\ell)}) = 1 - H_q(\varrho_i^{(n_\ell)}),$$

where the relative minimum distance satisfies

$$\varrho_i^{(n_\ell)} = \frac{\left\lceil \frac{d_E^2(L_\ell)}{d_E^2(L_i)} \right\rceil}{n_\ell}.$$

Let $\mathcal{C}^{(n_\ell)} := \left\{ C_i^{(n_\ell)} \right\}_{i=0}^{\ell-1}$. For each n_ℓ, define a packing

$$\mathcal{P}^{(n_\ell)} := \tau(\mathcal{C}^{(n_\ell)}) + L_\ell^{n_\ell}$$

as in Eq. (9).

Next we take ℓ increasingly to ∞. The following proposition describes the asymptotic density exponent of the packing family $\mathcal{F} = \{\mathcal{P}^{(n_\ell)}\}_{\ell \to \infty}$, where $n_\ell = \lceil q^{2\ell/m} |D_K|^{1/m} \rceil$ tends to ∞ as ℓ goes to ∞.

Proposition 3.7. *The asymptotic density exponent of the family $\mathcal{F}$ satisfies*

$$\begin{aligned} \lambda(\mathcal{F}) \quad \geqslant \quad & -1 - \frac{1}{2m} \log_2 |D_K| - \frac{1}{2} \log_2 \frac{m}{2\pi e} \\ & + \limsup_{\ell \to \infty} \left(\log_2 d_E(L_\ell) - \frac{1}{2} \log_2 n_\ell - \frac{\log_2 q}{m} \sum_{i=0}^{\ell-1} H_q' \left(\varrho_i^{(n_\ell)} \right) \right), \end{aligned} \tag{13}$$

where $H_q'(\varrho) = H_q(\varrho)$ for $0 < \varrho < \frac{q-1}{q}$ and $H_q'(\varrho) = 1$ for $\frac{q-1}{q} \leqslant \varrho \leqslant 1$.

Proof. By Corollary 3.5,

$$\begin{aligned}
\lambda(\mathcal{F}) &\geqslant \limsup_{\ell\to\infty} \frac{1}{mn_\ell} \log_2 \left(\frac{(d_E(L_\ell)/2)^{mn_\ell} V_{mn_\ell}}{\left(q^\ell \sqrt{|D_K|}\right)^{n_\ell}} \cdot \prod_{i=0}^{\ell-1} M_i^{(n_\ell)} \right) \\
&= -1 - \frac{1}{2m} \log_2 |D_K| + \\
&\quad + \limsup_{\ell\to\infty} \left(\log_2 d_E(L_\ell) - \frac{\ell}{m} \log_2 q + \frac{1}{mn_\ell} \log_2 V_{mn_\ell} + \sum_{i=0}^{\ell-1} \frac{\log_2 q \cdot k_i^{(n_\ell)}}{mn_\ell} \right).
\end{aligned}$$

By Eq. (3),

$$\begin{aligned}
& \limsup_{\ell\to\infty} \left(\frac{1}{mn_\ell} \log_2 V_{mn_\ell} + \sum_{i=0}^{\ell-1} \frac{\log_2 q \cdot k_i^{(n_\ell)}}{mn_\ell} \right) \\
= \;& \limsup_{\ell\to\infty} \left(-\frac{1}{2} \log_2 \frac{mn_\ell}{2\pi e} - \frac{1}{2mn_\ell} \log_2 mn_\ell \pi + \frac{\log_2 q}{m} \sum_{i=0}^{\ell-1} \frac{k_i^{(n_\ell)}}{n_\ell} \right) \\
\geqslant \;& -\frac{1}{2} \log_2 \frac{m}{2\pi e} + \limsup_{\ell\to\infty} \left(-\frac{1}{2} \log_2 n_\ell + \frac{\log_2 q}{m} \sum_{i=0}^{\ell-1} R_{GV}\left(q, \varrho_i^{(n_\ell)}\right) \right) \\
= \;& -\frac{1}{2} \log_2 \frac{m}{2\pi e} + \limsup_{\ell\to\infty} \left(-\frac{1}{2} \log_2 n_\ell + \frac{\log_2 q}{m} \sum_{i=0}^{\ell-1} \left(1 - H_q'\left(\varrho_i^{(n_\ell)}\right)\right) \right) \\
= \;& -\frac{1}{2} \log_2 \frac{m}{2\pi e} + \limsup_{\ell\to\infty} \left(\frac{\ell}{m} \log_2 q - \frac{1}{2} \log_2 n_\ell - \frac{\log_2 q}{m} \sum_{i=0}^{\ell-1} H_q'\left(\varrho_i^{(n_\ell)}\right) \right).
\end{aligned}$$

In summary, we get Eq. (13). □

3.4 Examples

We use some explicit examples to illustrate our new construction. Here Magma V2.21-4 [5, 4] will be employed to obtain the numerical results.

For the finite concatenation, we use the linear codes from [11] which have an explicit construction. For the asymptotic result, we set sufficiently large ℓ to approximate the limit.

Example 3.8. *Consider the number field $K = \mathbb{Q}[\alpha]$, where α is a root of the irreducible polynomial*

$$f(x) = x^4 - x^3 - x^2 + x + 1 \in \mathbb{Q}[x].$$

$K/\mathbb{Q}$ is a totally complex number field with extension degree $[K : \mathbb{Q}] = 4$ and absolute discriminant $|D_K| = 117$. The absolute discriminant is the smallest one of all totally complex number fields with degree 4 (see [16]).

(i) We consider a prime ideal $\mathfrak{p}$ lying over $3 \in \mathbb{Z}$. The Magma *code of this example is listed in Appendix A, while for other examples, the* Magma *code can be modified from this template.*

The key outputs of Appendix A are listed here:

```
The degree of K=Q[x]/(x^4 - x^3 - x^2 + x + 1) is m=4;
The absolute value of the discriminant of K is |d|=117;
```

```
P is a Prime Ideal of O
Two element generators:
[3, 0, 0, 0]
[2, 1, 1, 0] lying over 3 with norm q=9;
```

The above statement means $\mathfrak{p} = (3, 2 + \alpha + \alpha^2)$.

```
Finite Concatenation:
We can concatenate at most 3 linear codes to the lattice
 constructed by 64 copies of L_{P^3},
whose Hamming weights are required respectively at least
27;9;3;
```

Referring to [11], as the norm of $\mathfrak{p}$ *is* 9*, we can use the existing* 9*-ary linear codes* C_0, C_1, C_2 *with parameters*

$$[64, 25, 27]\,,\ [64, 49, 9]\,,\ [64, 61, 3]$$

respectively. The sum of the dimensions is 135*. Considering the concatenation in Corollary 3.5, we obtain a packing with dimension* $4 \times 64 = 256$*, whose center density* δ *satisfies*

```
Our packing is in dimension 256 with Log_2(center density)
 at least  208.088204168043224246772217517
```

Note that our packing is denser than the Barnes-Wall lattice BW_{256}*, whose density is listed in the table of dense sphere packings [8, Table 1.3].*

Using this ideal to construct a packing family as in Proposition 3.7, we get the following result.

```
Asymptotic result:
The asymptotic density exponent of the packing family is
 at least -1.442426720
```

Note that the above result means our packing family is asymptotically good.

(ii) *For this field, we similarly analyze other prime ideals and list part of our numerical results in Table 1, where* δ *is the center density and* λ *is the asymptotic density of the corresponding packing family.*

Example 3.9. *Consider the number field* $K = \mathbb{Q}[\alpha]$*, where* α *is a root of the irreducible polynomial*

$$f(x) = x^3 + x^2 - 2x - 1 \in \mathbb{Q}[x].$$

$K/\mathbb{Q}$ *is a totally real number field with extension degree* $[K : \mathbb{Q}] = 3$ *and absolute discriminant* $|D_K| = 49$*. The absolute discriminant is the smallest one of all totally real cubic number fields (see [16]). The numerical results on our examples are partially listed in Table 2.*

Table 1: Examples constructed from $\mathbb{Q}[x]/\left(x^4 - x^3 - x^2 + x + 1\right)$

$\mathfrak{p}$	q	dimension	$\log_2 \delta$ at least	λ at least
$(3, 2+\alpha+\alpha^2)$	9	180	108.52	-1.442
		256	208.09[1]	
		512	590.52	
$(7, 4+\alpha)$	7	256	190.63	-1.453
		400	410.15	

[1] The packing can be explicitly constructed with $\log_2 \delta$ larger than 192 given by the Barnes-Wall lattice BW_{256}.

Table 2: Examples constructed from $\mathbb{Q}[x]/\left(x^3 + x^2 - 2x - 1\right)$

$\mathfrak{p}$	q	dimension	$\log_2 \delta$ at least	λ at least
(2)	8	255	134.46	-1.628
$(7, 5+\alpha)$	7	192	83.68	-1.585
		255	157.63	

Example 3.10. *Consider the number field $K = \mathbb{Q}[\alpha]$, where α is a root of the irreducible polynomial*

$$f(x) = x^3 + x^2 - 1 \in \mathbb{Q}[x].$$

$K/\mathbb{Q}$ is a number field with extension degree $[K : \mathbb{Q}] = 3$ and absolute discriminant $|D_K| = 23$. The absolute discriminant is the smallest one of all cubic number fields (see [16]). The numerical results on our examples are partially listed in Table 3.

Table 3: Examples constructed from $\mathbb{Q}[x]/\left(x^3 + x^2 - 1\right)$

$\mathfrak{p}$	q	dimension	$\log_2 \delta$ at least	λ at least
(2)	8	96	24.70	-1.429
		192	115.40	
$(5, 2+\alpha)$	5	150	69.47	-1.445
		180	101.01	
$(7, 11+\alpha)$	7	255	187.32	-1.430

Example 3.11. *Consider the number field $K = \mathbb{Q}[\alpha]$, where α is a root of the irreducible polynomial*

$$f(x) = x^6 + x^3 + 1 \in \mathbb{Q}[x].$$

$K/\mathbb{Q}$ is a number field with extension degree $[K : \mathbb{Q}] = 6$ and absolute discriminant $|D_K| = 19683$. The polynomial $f(x) = x^6 + x^3 + 1$ is the 9th cyclotomic polynomial over $\mathbb{Q}$. The numerical results on our examples are partially listed in Table 4.

4 Conclusion

This paper provides a new method to construct dense packings using canonical $\mathbb{Q}$-embedding of algebraic number fields, and special codes over the residue field of some prime ideals. With the help of Magma V2.21-4,

Table 4: Examples constructed from $\mathbb{Q}[x]/\left(x^6+x^3+1\right)$

$\mathfrak{p}$	q	dimension	$\log_2 \delta$ at least	λ at least
$(3, 2+\alpha)$	3	180	109.71	-1.868
		192	122.72	

several explicit constructions were provided in Examples 3.8 - 3.11. Especially, in $\mathbb{R}^{256}$, a packing denser than the Barnes-Wall lattice BW_{256} was obtained. Moreover, this method can be utilized to construct asymptotically good packing families. For each number field and prime ideal in Tables 1 - 4, a lower bound on the asymptotic density exponent of the corresponding packing family was provided.

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A Magma Code for Example 3.8

```
R:=RealField(10); Q:=RationalField();
W<x>:=PolynomialRing(Q); f:=x^4-x^3-x^2+x+1;
            /*Input a polynomial over rational number field Q*/
if IsIrreducible(f) then   /* Test whether f is irreducible*/
 K<a>:=NumberField([f]); O:=MaximalOrder(K); m:=AbsoluteDegree(K);
 printf"The degree of K=Q[x]/(%o) is m=%o;\n",f,m;
 d:=AbsoluteDiscriminant(K);
 printf"The absolute value of the discriminant of K \
  is |d|=%o;\n",d;

 p:=3;     /*Choose p=3 as a base prime number*/
 J:=Decomposition(O,p); P:=J[1,1]; q:=Norm(P);
 printf"P is a %o lying over %o with norm q=%o;\n",P,p,q;

 printf"\nFinite Concatenation:\n";
 n:=64; l:=Floor(m/2*Log(n)/Log(q));
        /*Set the length of the codes*/
 L:=MinkowskiLattice(P^l); b:=Minimum(L);
 printf"We can concatenate at most %o linear codes to \
  the lattice constructed by %o copies of L_{P^%o},\n \
  whose Hamming weights are required respectively at least \
  \n",l,n,l;

 for t in [0..l-1] do
   Z:=MinkowskiLattice(P^t); h:=Ceiling(Minimum(L)/Minimum(Z));
   printf"%o;",h;
 end for;
 printf"\n";
```

```
 printf"Refer to the Codetable.de to get the largest dimension \
  of the corresponding linear codes with length %o; \n",n;
 fromtable:=135;
 printf"Here the sum of dimensions T=%o;\n",fromtable;
         /*For this example, the sum of the dimensions is 135*/
 c:=(m*n*Log(b^0.5)-n*l*Log(q)-0.5*n*Log(d))/Log(2)-m*n;
 printf"Our packing is in dimension %o with \
  Log_2(center density) at least\n %o\n",n*m,\
  c+Log(q)/Log(2)*fromtable;

 printf"\nAsymptotic result:\n";
 l:=1000; n:=Ceiling(q^(2*l/m)*d^(1/m));
         /*Set l sufficiently large to approximate the limit*/
 L:=MinkowskiLattice(P^l);b:=Minimum(L);
 Sum:=0;
 for t in [l-1..0 by -1] do
   Z:=MinkowskiLattice(P^t);
   g:=Ceiling(Minimum(L)/Minimum(Z));
   varrho:=g/n;
    if varrho le (q-1)/q then
      Sum:=Sum+(varrho*Log(q-1)-varrho*Log(varrho)\
       -(1-varrho)*Log(1-varrho))/Log(q);
    else Sum:=Sum+1;
    end if;
 end for;

 Lambda:=-1-1/(2*m)*Log(d)/Log(2)-0.5*Log(m/(2*Pi(R)*Exp(1))) \
  /Log(2)+Log(b^0.5)/Log(2)-0.5*Log(n)/Log(2) \
  -1/m*Log(q)/Log(2)*Sum;
 printf"The asymptotic density exponent of the packing family is \
 \n at least %o",Lambda;

else
printf"The polynomial %o is not irreducible over Q.\n",f;
end if;
```

References

[1] M.F. Atiyah and I.G. MacDonald. *Introduction to Commutative Algebra*. Addison-Wesley Series in Mathematics, Westview Press, 1994.

[2] C. Bachoc. *Applications of Coding Theory to the Construction of Modular Lattices* . Journal of Combinatorial Theory, Series A, 1997, **78**, no. 1: 92 – 119.

[3] E. Bayer-Fluckiger. *Ideal Lattices*. A Panorama of Number Theory Or The View from Baker's Garden, Cambridge University Press, 2002, pp. 168–184.

[4] W. Bosma, J.J. Cannon, C. Fieker, and A. Steel (eds.). *Handbook of Magma Functions*, v2.21 ed., online http://magma.maths.usyd.edu.au/magma/handbook/, 2015.

[5] W. Bosma, J.J. Cannon, and C. Playoust. *The Magma Algebra System. I. The User Language*. J. Symbolic Comput, 1997, **24**, no. 3-4, 235–265, Computational algebra and number theory (London, 1993).

[6] J.W.S. Cassels. *An Introduction to the Geometry of Numbers*. Springer-Verlag, New York, 1997.

[7] S. Cheng. *Improvement on Asymptotic Density of Packing Families Derived from Multiplicative Lattices*. Finite Fields and Their Applications, 2015, **36**: 133–150

[8] J.H. Conway and N.J.A. Sloane. *Sphere Packings, Lattices and Groups*, 3rd ed., Springer-Verlag, New York, 1999.

[9] N.D. Elkies. Lattices, Linear Codes, and Invariants, Part II. Notices of the AMS 47(11), 1382–1391 (2000)

[10] P. Gaborit and G. Zémor. *On the Construction of Dense Lattices with A Given Automorphisms Group*. Annales de l'institut Fourier, 2007, **57**, no. 4: 1051-1062

[11] M. Grassl. *Bounds on the Minimum Distance of Linear Codes and Quantum Codes*, Online available at http://www.codetables.de, 2007, Accessed on 2016-06-25.

[12] F.J. MacWilliams and N.J.A. Sloane. *The Theory of Error-correcting Codes*, North-Holland Mathematical Library, North-Holland Publishing Company, 1977.

[13] D. Micciancio and S. Goldwasser. *Complexity of Lattice Problems: A Cryptographic Perspective*, The Kluwer International Series in Engineering and Computer Science, vol. 671, Kluwer Academic Publishers, Boston, Massachusetts, March 2002.

[14] J. Neukirch. *Algebraic Number Theory*, Grundlehren der mathematischen Wissenschaften : a series of comprehensive studies in mathematics, vol. 322, Springer-Verlag Berlin Heidelberg, 1999.

[15] P.Q. Nguyen. *Hermite's Constant and Lattice Algorithms*, The LLL Algorithm - Survey and Applications (P.Q. Nguyen and B. Vallée, eds.), Information Security and Cryptography, Springer, 2010, pp. 19–69.

[16] A. M. Odlyzko. *Bounds for Discriminants and Related Estimates for Class Numbers, Regulators and Zeros of Zeta Functions : A Survey of Recent Results*, Journal de théorie des nombres de Bordeaux, 1990, **2**, no. 1: 119–141.

[17] J.A. Rush. *A Lower Bound on Packing Density*. Inventiones mathematicae, 1989, **98**, no. 3: 499-509.

[18] M.A. Tsfasman and S.G. Vlăduţ. *Algebraic-Geometric Codes*, Kluwer Academic, 1991.

[19] C. Xing. *Dense Packings from Quadratic Fields and Codes*. Journal of Combinatorial Theory, Series A, 2008, **115**, no. 6: 1021–1035.

[20] C. Zong. *Sphere Packings*, Universitext (1979), Springer-Verlag New York, 1999.